Extremum Seeking for Systems Described by Partial Differential Equations and Its Applications

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São Carlos – SP , DINCON 2019

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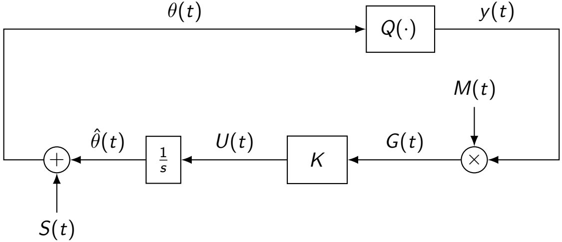
OUTLINE:

Gradient Extremum Seeking



Assumption: quadratic static map

$$y(t) = y^* + \frac{H}{2}(\theta - \theta^*)^2$$

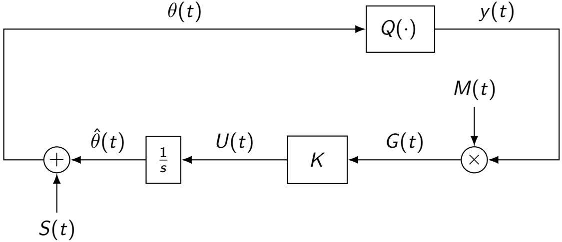
Gradient estimate:

$$G(t) = \frac{2}{a}\sin(\omega t)y(t)$$

Perturbation signals:

$$S(t) = a\sin(\omega t)$$
$$M(t) = \frac{2}{a}\sin(\omega t)$$

Gradient Extremum Seeking



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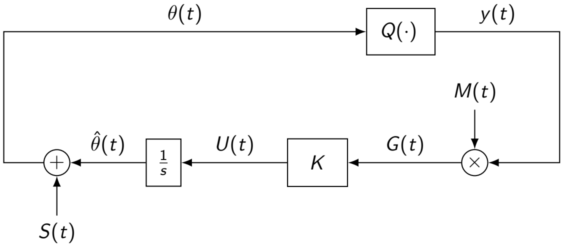

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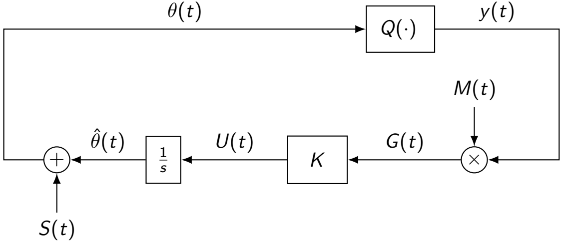
Gradient Extremum Seeking



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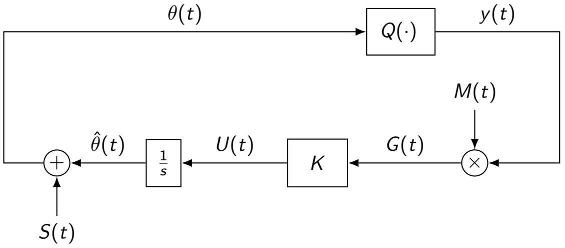
$$= KM(t)\left[y^* + \frac{H}{2}(\tilde{\theta}(t) + a\sin(\omega t))^2\right]$$

Gradient Extremum Seeking



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$$= KM(t)\left[y^* + \frac{H}{2}(\tilde{\theta}(t) + a\sin(\omega t))^2\right]$$

Average error dynamics:

Recap: Averaging

Consider the original system

$$\dot{z} = f(\omega t, z), \quad z(0) = z_0, \tag{1}$$

and the average system

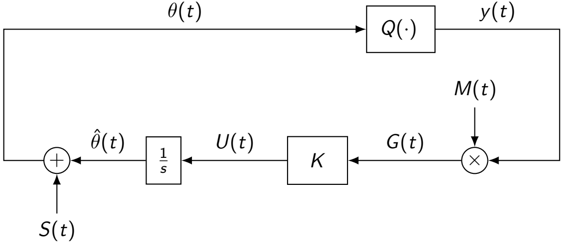
$$\dot{z}_{\mathrm{av}} = f_{\mathrm{av}}(z_{\mathrm{av}}), \quad z_{\mathrm{av}}(0) = z_{\mathrm{av},0}, \quad f_{\mathrm{av}}(z_{\mathrm{av}}) = \frac{1}{T}\int_0^T f(\tau, z_{\mathrm{av}})d\tau.$$

If $z_{\mathrm{av}} = 0$ is an exponentially stable solution, then there exists $\bar{\omega} > 0$ such that for all $\omega > \bar{\omega}$

$$\|z(t) - z_{\mathrm{av}}(t)\| \leq \mathcal{O}(1/\omega), \quad \forall\ t \in [0, \infty),$$

Furthermore, (1) has a unique exponentially stable, T-periodic solution $\bar{z}(t, 1/\omega)$ with the property $\|\bar{z}(t, 1/\omega)\| \leq \mathcal{O}(1/\omega)$.

Gradient Extremum Seeking



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Perturbation signal:

$$S(t) = a\sin(\omega t)$$

Estimation error: $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*(\mathrm{t})$

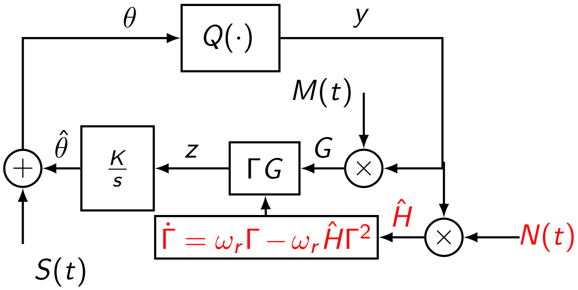
Error dynamics: $\dot{\tilde{\theta}}(t) = KM(t)y(t)$

$$= KM(t)\left[y^* + \frac{H}{2}(\tilde{\theta}(t) + a\sin(\omega t))^2\right]$$

Average error dynamics: $\bar{K} = KH$

$$\dot{\tilde{\theta}}_{\rm av}(t) = \bar{K}\tilde{\theta}_{\rm av}(t)$$

Newton-based Extremum Seeking



Hessian estimate:

$$\hat{H}(t) = N(t)y(t)$$

Demodulating Signal:

$$N(t) = -\frac{8}{a^2}\cos(2\omega t)$$

Auxiliary Variable:

$$z(t) = \Gamma(t)G(t)$$

Riccati Filter (error):

$$\tilde{\Gamma}_{\mathrm{av}}(t) = \Gamma_{\mathrm{av}}(t) - H^{-1}$$

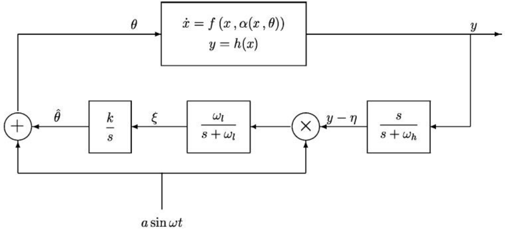
Average error dynamics: $\bar{K} = -K < 0$

$$\hat{H}_{\mathrm{av}}(t) = H$$

$$\Gamma_{\mathrm{av}}(t) \to H^{-1}$$

$$\dot{\tilde{\theta}}_{\mathrm{av}}(t) = \bar{K}\tilde{\theta}_{\mathrm{av}}(t)$$

Gradient Extremum Seeking for Dynamic (ODE) Systems

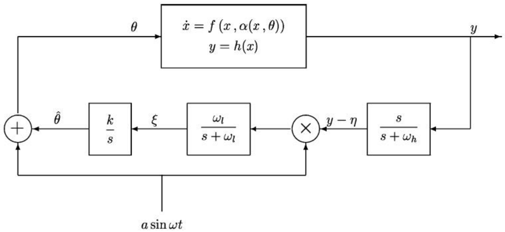


Assumptions:

- $f(t, \alpha(x, \theta)) = 0$ if and only if $x = l(\theta)$.
- For each $\theta \in \mathbb{R}$, the equilibrium $x = l(\theta)$ of the system is locally exponentially stable with decay and overshoot constants uniform in θ.
- $y = Q(\theta) = h \circ l(\theta)$.
- There exists $\theta^* \in \mathbb{R}$ such that $(h \circ l)'(\theta^*) = 0$ and $(h \circ l)'(\theta^*) < 0$.

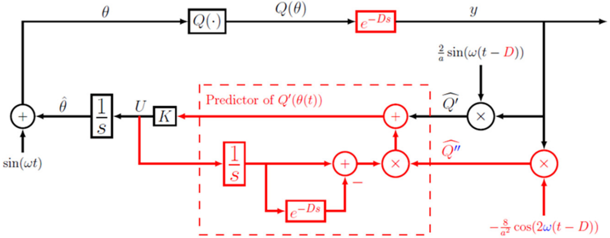
Analysis Tools: **Averaging** and **Singular Perturbation**

Gradient Extremum Seeking for Dynamic (ODE) Systems



- **Theorem 1:** There exists a ball of initial conditions around the point $(x, \hat{\theta}, \xi, \eta) = (l(\theta^*), \theta^*, 0, h \circ l(\theta^*))$ and constants $\bar{\omega}, \bar{\delta}$ and $\bar{a}$ such that for all $\omega \in (0, \bar{\omega})$, $\delta \in (0, \bar{\delta})$, and $a \in (0, \bar{a})$, the solution $(x(t), \hat{\theta}(t), \xi(t), \eta(t))$ exponentially converges to an $O(\omega + \delta + a)$-neighborhood of that point. Furthermore, $y(t)$ converges to an $O(\omega + \delta + a)$-neighborhood of $h \circ l(\theta^*)$.

Predictor Feedback for ES with Sensor Delays



Delayed Output:

$$y(t) = Q(\theta(t - D))$$

Gradient Estimate:

$$\frac{1}{\Pi}\int_0^t M(\sigma) y d\sigma = H\tilde{\theta}_{av}$$

$$\hat{Q}'_{av} = (My)_{av} = H\tilde{\theta}_{av}(t - D)$$

Hessian Estimate:

$$\frac{1}{\Pi}\int_0^t N(\sigma) y d\sigma = H$$

$$\hat{Q}''_{av} = (Ny)_{av} = H$$

Dither and Demodulation Signals:

$$S(t) = a\sin(\omega t)$$

$$M(t) = \frac{2}{a}\sin(\omega(t - D))$$

$$N(t) = -\frac{8}{a^2}\cos(2\omega(t - D))$$

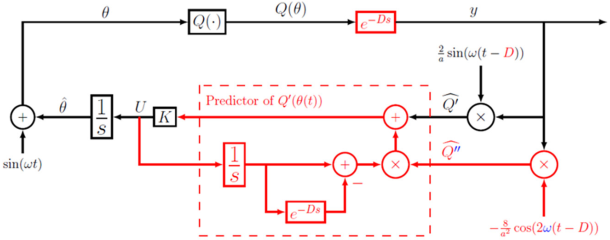
Averaging Analysis (without prediction):

$$U(t) = K\hat{Q}'(t)$$

$$\dot{\hat{\theta}}(t) = U(t)\,, \quad \dot{\tilde{\theta}}_{av} = kH\tilde{\theta}_{av}(t - D)$$

$$\dot{\hat{Q}}'_{av}(t) = HU_{av}(t - D)$$

Predictor Feedback for ES with Sensor Delays



Delay Prediction Feedback:

$$U_{av}(t) = K\hat{Q}'_{av}(t+D)$$

Future State:

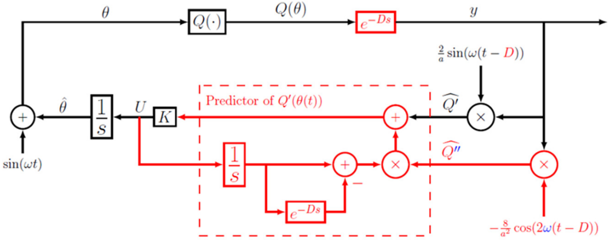
$$\hat{Q}'_{av}(t+D) = \hat{Q}'_{av}(t) + H\int_{t-D}^{t} U_{av}(\sigma)d\sigma$$

Filtered Predictor Feedback Law:

$$U(t) = \frac{c}{s+c}\left\{K\left[\hat{Q}'(t) + \hat{Q}''(t)\int_{t-D}^{t} U(\tau)d\tau\right]\right\}$$

Lag Filter: Hale and Lunel's Averaging Theorem + Inverse optimality

Predictor Feedback for ES with Sensor Delays



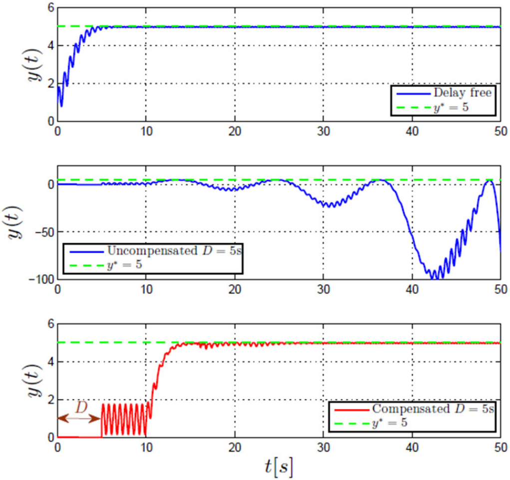
- **Theorem:** Consider the control system in the Figure with delayed output and $D \geq 0$ being a simple scalar. There exist $c^* > 0$ such that, $\forall c \geq c^*$, $\omega^* > 0$ such that, $\forall \omega > \omega^*$, the closed-loop delayed system with state $\tilde{\theta}(t-D)$, $U(\sigma)$, $\forall \sigma \in [t-D, t]$, has a unique exponentially stable periodic solution in t of period $\Pi := 2\pi/\omega$, denoted by $\tilde{\theta}^{\Pi}(t-D)$, $U^{\Pi}(\sigma)$, $\forall \sigma \in [t-D, t]$, satisfying, $\forall \geq 0$:

$$\left(\left|\tilde{\theta}^{\Pi}(t-D)\right|^2 + \left|U^{\Pi}(t-D)\right|^2 + \int_{t-D}^{t} \left|U^{\Pi}(\tau)\right|^2 d\tau\right)^{1/2} \leq \mathcal{O}(1/\omega)\,.$$

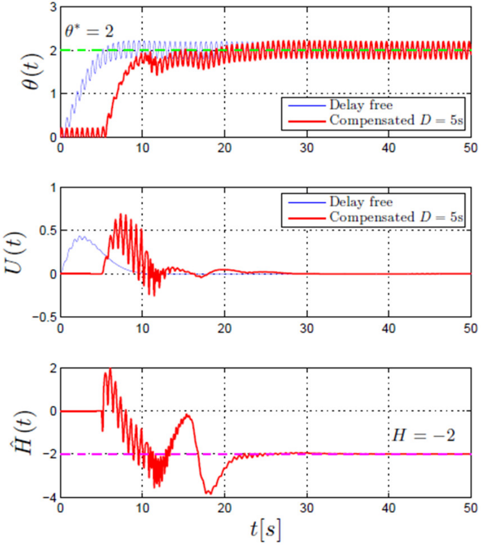
Furthermore, $\lim_{t\to+\infty} \sup |\theta(t) - \theta^*| = \mathcal{O}(a + 1/\omega)$ and $\lim_{t\to+\infty} \sup |y(t) - y^*| = \mathcal{O}(a^2 + 1/\omega^2)$.

Predictor Feedback for ES with Delays - Simulation

$Q(\theta) = 5 - (\theta - 2)^2$, $(\theta^*, y^*) = (2, 5)$, $H = -2$ and $D = 5s$



Predictor Feedback for ES with Delays - Simulation



Predictor Feedback for ES with Delays - Application

Neuromuscular Electrical Stimulation (NMES)

Challenges for Modeling and Actuation

Patients Variability

- Different kinds of lesion (parametric/relative degree uncertainties)
- Patient response changes over time (time-varying system)
- Saturation, dead-zone and fatigue (nonlinear phenomena)
- Time delays (small but present)
- Gravity action (disturbances for upperward movements)
- Hybrid bidirectional actuator (biceps and triceps)

Predictor Feedback for ES with Delays - Application

Assistive Robotics for Stroke Patients

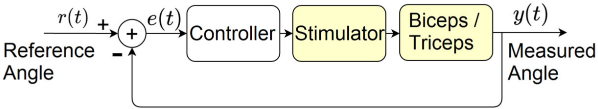
- Motor disorder
- Spasticity (hypertonia)
- Physiotherapy

Rehabilitation

- Passive or active movement
- Closed-loop feedback aids patients' recovery
- Design control laws for NMES

Predictor Feedback for ES with Delays - Application

Adaptive Control Strategy



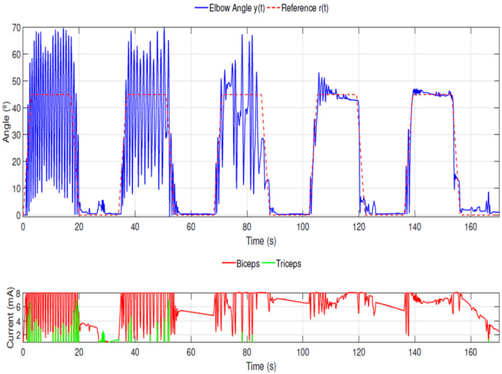
Which Controller? Adaptive Control!

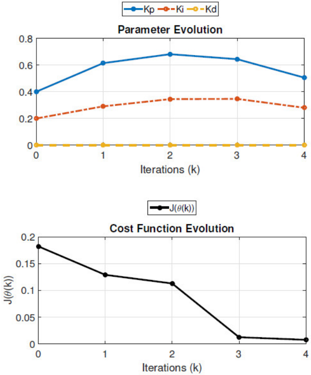
- Conventional Adaptive Control (control parametrization)
- Model Reference Adaptive Control (delays/relative degree obstacles)
- PID with Extremum Seeking for adaptation: $J(\theta) = \frac{1}{T-t_0}\int_{t_0}^{T} e^2(t)dt$

Automatic controller tuning: adaptation

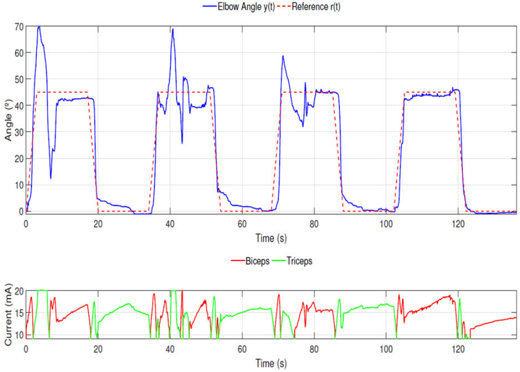
- May solve the huge gap between healthy volunteers and stroke patients

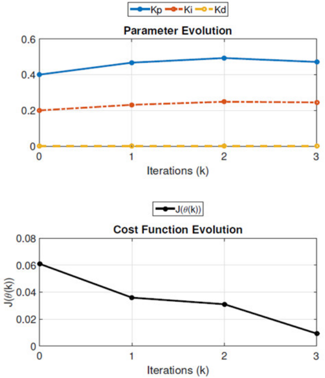
Predictor Feedback for ES with Delays - Experiment 1



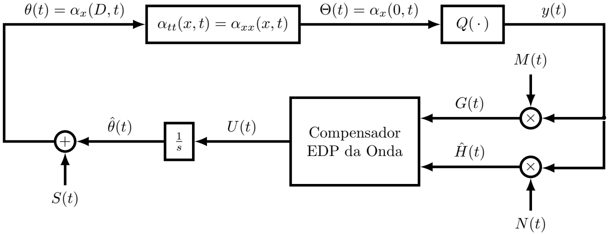


Predictor Feedback for ES with Delays - Experiment 2





PDE Compensation for ES with Wave Process



Actuation dynamics (Wave):

$$\Theta(t) = \partial_x \alpha(0,t)$$
$$\partial_{tt}\alpha(x,t) = \partial_{xx}\alpha(x,t), \quad x \in [0,D]$$
$$\alpha(0,t) = 0$$
$$\partial_x \alpha(D,t) = \theta(t)$$

Output map:

$$y(t) = Q(\Theta) = y^* + \frac{H}{2}(\Theta(t) - \Theta^*)^2$$

Dither Signals:

$$S(t) = \frac{a}{\omega}\sin(\omega D)\sin(\omega t)$$
$$M(t) = \frac{2}{a}\sin(\omega t)$$
$$N(t) = -\frac{8}{a^2}\cos(2\omega t)$$

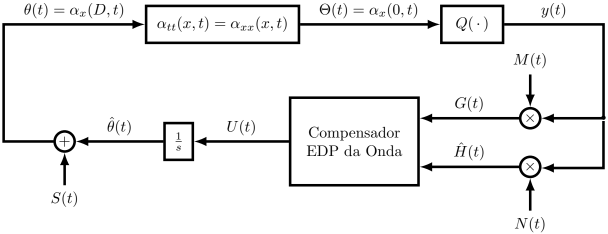
Estimate Error:

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$$

Propagated Estimate Error:

$$\vartheta(t) = \hat{\Theta}(t) - \Theta^*$$

PDE Compensation for ES with Wave Process



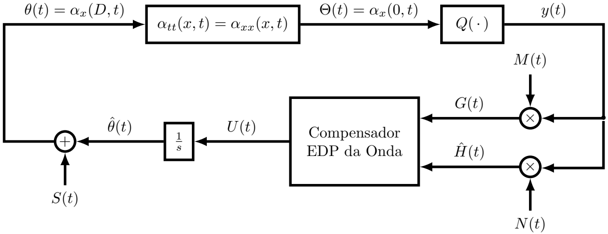
Estimated Error Dynamics:

$$\dot{\vartheta}(t) = \partial_x u(0,t)$$
$$\partial_{tt} u(x,t) = \partial_{xx} u(x,t), \quad x \in [0,D]$$
$$u(0,t) = 0$$
$$\partial_x u(D,t) = U(t)$$

Control Law with Wave Process Compensation:

$$U(t) = \tfrac{c}{s+c}\left\{\bar{c}\left[K\hat{H}(t)u(D,t) - \partial_t u(D,t)\right] + \bar{\rho}(D)KG(t) + K\hat{H}(t)\int_0^D \bar{\rho}(D-\sigma)\partial_t u(\sigma,t)d\sigma\right\}$$

PDE Compensation for ES with Wave Process



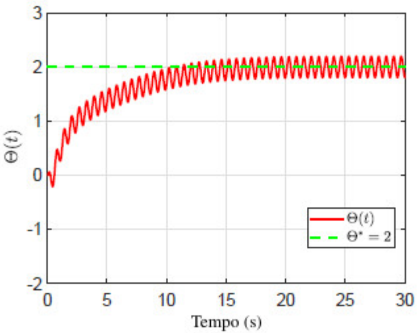
Theorem: Consider the closed-loop system in the figure above. For a sufficiently large $c > 0$, there exists some $\bar{\omega}(c) > 0$, such that $\forall \omega > \bar{\omega}$, the closed-loop system with states $\vartheta(t)$, $u(x,t)$, has a unique exponentially stable periodic solution in t of period $\Pi := 2\pi/\omega$, denoted by $\vartheta^{\Pi}(t)$, $u^{\Pi}(x,t)$, satisfying $\forall t \geq 0$:

$$\left(\left|\vartheta^{\Pi}(t)\right|^2 + \left\|\partial_x u^{\Pi}(t)\right\|^2 + \left\|\partial_t u^{\Pi}(t)\right\|^2 + \left|\partial_x u^{\Pi}(D,t)\right|^2\right)^{1/2} \leq \mathcal{O}\left(1/\omega\right).$$

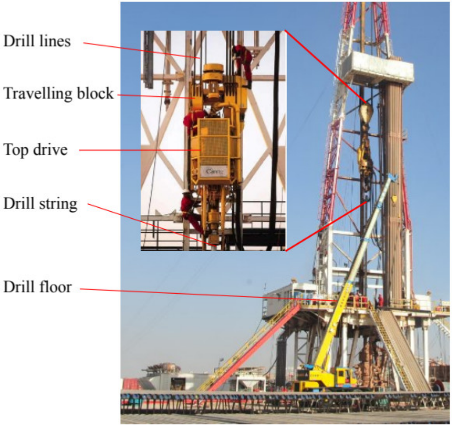
Furthermore, $\limsup_{t\to\infty} |\theta(t) - \theta^*| = \mathcal{O}\left(a/\omega + 1/\omega\right)$, $\limsup_{t\to\infty} |\Theta(t) - \Theta^*| = \mathcal{O}\left(a + 1/\omega\right)$ and $\limsup_{t\to\infty} |y(t) - y^*| = \mathcal{O}\left(a + 1/\omega^2\right)$.

PDE Compensation for ES with Wave Process - Simulation

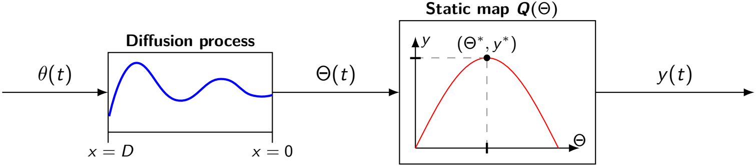
$H = -0.2,\ \Theta^* = 2,\ y^* = 5,\ D = 1,\ \omega = 10,\ a = 0.2,\ c = 10,$
$\bar{c} = 0.5$ and $K = 0.4$



PDE Compensation for ES with Wave Process - Application



Problem Statement and Motivation

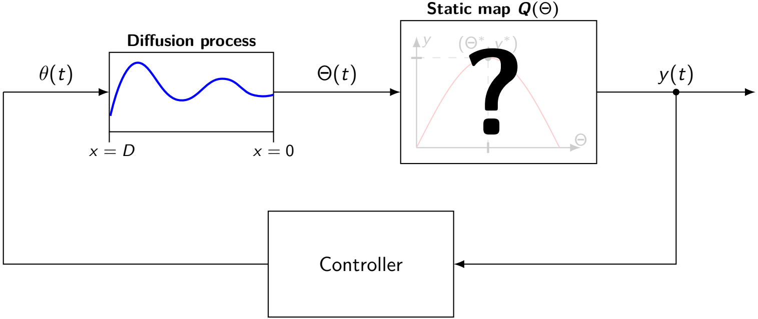


- heat transfer
- mixing of chemical substances
- information distribution

Signals

- $\theta(t)$: input/actuator
- $y(t)$: output

Problem Statement and Motivation

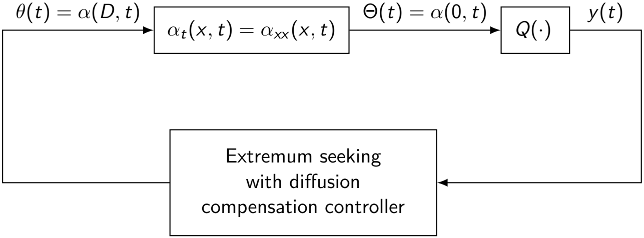


Semi-model-based control concept

System Signals

- $\theta(t)$: input/actuator
- $y(t)$: output

Problem Statement and Motivation



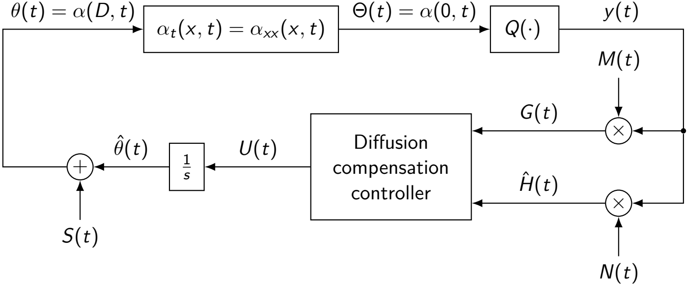
Assumptions

- known actuator dynamics
- unknown static map
- existence of extremum (max)

Questions

- controller design
- stability
- convergence

Extremum Seeking Control Loop – Gradient Case*



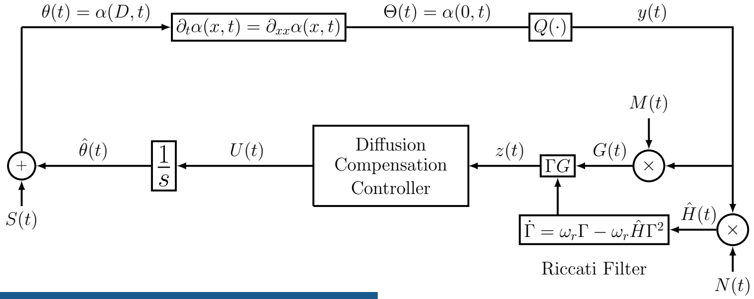
Two possibilities for diffusion compensation:

$$U(t) = \frac{c}{s+c}\left\{K\left[G(t) + \hat{H}(t)\int_0^D (D-r)u(r,t)dr\right]\right\},$$

$$U(t) = \frac{c}{s+c}\left\{K\left[G(t) + \hat{H}(t)\left(\hat{\theta}(t) - \Theta(t) + a\sin(\omega t)\right)\right]\right\}$$

* J. Feiling, S. Koga, M. Krstic, and T. R. Oliveira. "Gradient extremum seeking for static maps with actuation dynamics governed by diffusion PDEs". In: Automatica 95.7 (2018), pp. 197–206.

Extremum Seeking Control Loop – Newton Case



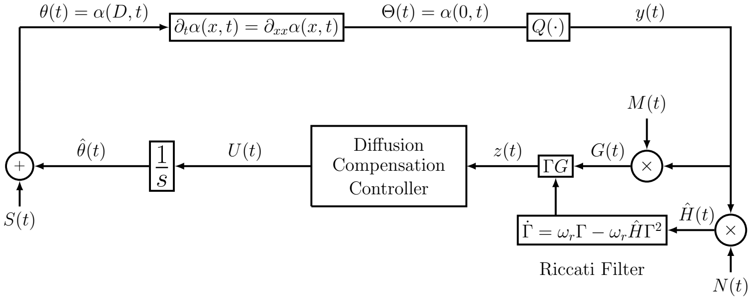
Actuation Dynamics

$$\Theta(t) = \alpha(0,t)$$
$$\alpha_t(x,t) = \alpha_{xx}(x,t), \quad x \in [0,D]$$
$$\alpha_x(0,t) = 0$$
$$\alpha(D,t) = \theta(t)$$

Adaption

- Perturbation Signal $S(t)$
- Hessian estimate
- Error dynamics

Extremum Seeking Control Loop – Newton Case



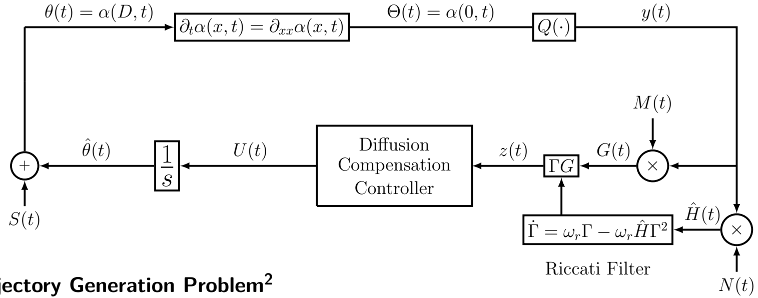
Hessian Estimate[1]

$$\hat{H}(t) = N(t)y(t)$$

$$N(t) = -\frac{8}{a}\cos(2\omega t)$$

[1] A. Ghaffari, M. Krstić, and D. Nešic. "Multivariable Newton-based extremum seeking". Automatica 48.8 (2012).

Extremum Seeking Control Loop – Newton Case



Trajectory Generation Problem[2]

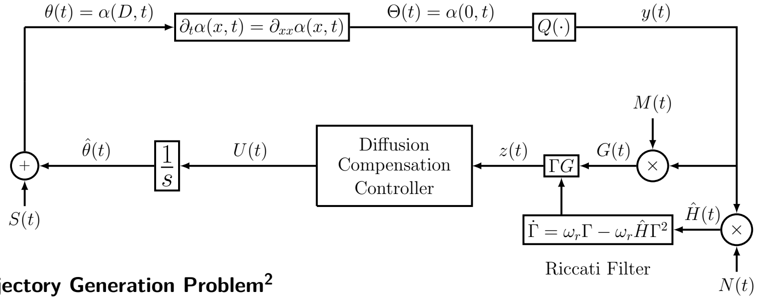
Perturbation Signal

$$S(t) = \beta(D,t)$$
$$\beta_t(x,t) = \beta_{xx}(x,t), \quad x \in [0,D]$$
$$\beta_x(0,t) = 0$$
$$\beta(0,t) = a\sin(\omega t)$$

$$S(t) = \frac{1}{2}ae^{\sqrt{\frac{\omega}{2}}D}\sin\left(\omega t + \sqrt{\frac{\omega}{2}}D\right) + \frac{1}{2}ae^{-\sqrt{\frac{\omega}{2}}D}\sin\left(\omega t - \sqrt{\frac{\omega}{2}}D\right)$$

[2] M. Krstić and A. Smyshlyaev. "Boundary control of PDEs: A course on backstepping designs". Vol. 16. Siam, 2008.

Extremum Seeking Control Loop – Newton Case



Trajectory Generation Problem[2]

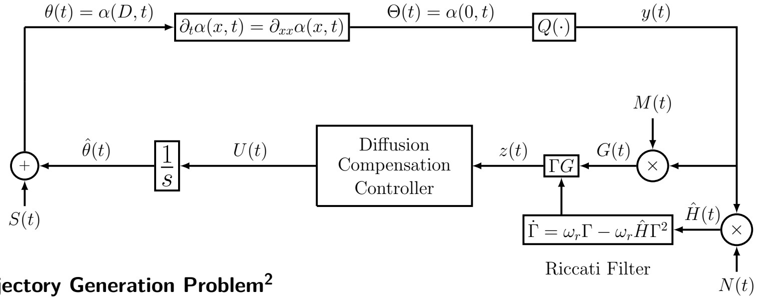
Perturbation Signal

$$S(t) = \beta(D,t)$$
$$\beta_t(x,t) = \beta_{xx}(x,t), \quad x \in [0,D]$$
$$\beta_x(0,t) = 0$$
$$\beta(0,t) = a\sin(\omega t)$$

$$S(t) = \frac{1}{2} a e^{\sqrt{\frac{\omega}{2}}D} \sin\left(\omega t + \sqrt{\frac{\omega}{2}}D\right) + \frac{1}{2} a e^{-\sqrt{\frac{\omega}{2}}D} \sin\left(\omega t - \sqrt{\frac{\omega}{2}}D\right)$$

[2] M. Krstić and A. Smyshlyaev. "Boundary control of PDEs: A course on backstepping designs". Vol. 16. Siam, 2008.

Extremum Seeking Control Loop – Newton Case



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$$\beta(0,t) = a\sin(\omega t)$$

Hessian Estimate[1]

$$\hat{H}(t) = N(t)y(t)$$
$$N(t) = -\frac{8}{a}\cos(2\omega t)$$

Extremum Seeking Control Loop - Error Dynamics

Estimation Error: $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$

Error dynamics: $\dot{\tilde{\theta}}(t) = \dot{\hat{\theta}}(t) = U(t)$

Extremum Seeking Control Loop - Error Dynamics

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Propagated error: $\vartheta(t) = \hat{\Theta}(t) - \theta^*$

$$\vartheta(t) := \bar{\alpha}(0, t)$$

$$\bar{\alpha}_t(x, t) = \bar{\alpha}_{xx}(x, t), \quad x \in [0, D]$$

$$\bar{\alpha}_x(0, t) = 0$$

$$\bar{\alpha}(D, t) = \tilde{\theta}(t)$$

Extremum Seeking Control Loop - Error Dynamics

Estimation Error: $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$

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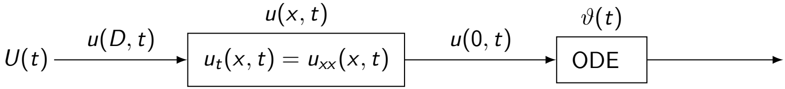
$$\bar{\alpha}_x(0,t) = 0$$
$$\bar{\alpha}(D,t) = \tilde{\theta}(t)$$

Propagated error dynamics:

$$\dot{\vartheta}(t) = u(0,t)$$
$$u_t(x,t) = u_{xx}(x,t), \quad x \in [0,D]$$
$$u_x(0,t) = 0$$
$$u(D,t) = U(t)$$



Diffusion Compensation Controller

Propagated error dynamics:

$$\dot{\vartheta}(t) = u(0,t)$$
$$u_t(x,t) = u_{xx}(x,t), \quad x \in [0,D]$$
$$u_x(0,t) = 0$$
$$u(D,t) = U(t)$$

Target system:

$$\dot{\vartheta}(t) = \bar{K}\vartheta(t) + w(0,t), \quad \bar{K} < 0$$
$$w_t(x,t) = w_{xx}(x,t), \quad x \in [0,D]$$
$$w_x(0,t) = 0$$
$$w(D,t) = 0$$

Backstepping transformation[3]:

$$w(x,t) = u(x,t) - \int_0^x q(x,r)u(r,t)dr - \gamma(x)\vartheta(t)$$
$$q(x,r) = \bar{K}(x-r), \qquad \gamma(x) = \bar{K}$$

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

[3] M. Krstić. "Compensating actuator and sensor dynamics governed by diffusion PDEs". In: Systems & Control Letters 58.5 (2009), pp. 372–377.

Diffusion Compensation Controller

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

Average-based estimates

$$G_{av}(t) = [N(t)y(t)]_{av} = H\vartheta_{av}(t), \qquad \hat{H}_{av} = H$$

Diffusion Compensation Controller

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

Average-based estimates

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Averaged control law:

$$U_{av}(t) = \bar{K}\vartheta_{av}(t) + \bar{K}\int_0^D (D-r)u_{av}(r,t)dr$$

Diffusion Compensation Controller

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

Average-based estimates

$$G_{\rm av}(t) = [N(t)y(t)]_{\rm av} = H\vartheta_{\rm av}(t), \qquad \hat{H}_{\rm av} = H$$

Averaged control law: $\bar{K} = -K,\ K > 0$

$$U_{\rm av}(t) = -K\vartheta_{\rm av}(t) - K\int_0^D (D-r)u_{\rm av}(r,t)dr$$

Diffusion Compensation Controller

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

Average-based estimates

$$G_{av}(t) = [N(t)y(t)]_{av} = H\vartheta_{av}(t), \qquad \hat{H}_{av} = H$$

Averaged control law:

$z_{\rm av}(t) = \vartheta_{\rm av}(t) + \tilde{\Gamma}_{\rm av}(t)H\vartheta_{\rm av}(t)$... linearization at $\tilde{\Gamma}_{\rm av} = 0$

$$U_{av}(t) = -Kz_{av}(t) - K\int_0^D (D-r)u_{av}(r,t)dr$$

Diffusion Compensation Controller

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

Average-based estimates

$$G_{av}(t) = [N(t)y(t)]_{av} = H\vartheta_{av}(t), \qquad \hat{H}_{av} = H$$

Averaged control law:

$$U_{av}(t) = -Kz_{av}(t) - K\int_0^D (D-r)u_{av}(r,t)dr$$

Averaged-based control law: $c > 0$ and large!

$$U(t) = -K\left[z(t) + \int_0^D (D-r)u(r,t)dr\right]$$

Diffusion Compensation Controller

Control law:

$$U(t) = \bar{K}\vartheta(t) + \bar{K}\int_0^D (D-r)u(r,t)dr$$

Average-based estimates

$$G_{av}(t) = [N(t)y(t)]_{av} = H\vartheta_{av}(t), \qquad \hat{H}_{av} = H$$

Averaged control law:

$$U_{av}(t) = -Kz_{av}(t) - K\int_0^D (D-r)u_{av}(r,t)dr$$

Averaged-based control law: $c > 0$ and large!

$$U(t) = \frac{c}{s+c}\left\{-K\left[z(t) + \int_0^D (D-r)u(r,t)dr\right]\right\}$$

Closed-loop Stability

Closed-loop system:

$$\dot{\vartheta}(t) = u(0,t)$$
$$u_t(x,t) = u_{xx}(x,t), \quad x \in [0,D]$$
$$u_x(0,t) = 0$$
$$u(D,t) = U(t)$$
$$\dot{U}(t) = -cU(t) - cK\left[z(t) + \int_0^D (D-r)u(r,t)dr\right]$$

Statement

- (ϑ, u) exponentially stable in $\mathcal{H}_1$
- $(\theta(t), \Theta(t), y(t))$ converge to a neighborhood of $(\theta^*, \Theta^*, y^*)$

Closed-loop Stability

Stability & Convergence Theorem

Consider the closed-loop system. For a sufficiently large $c > 0$, there exists some $\bar{\omega}(c) > 0$, such that $\forall \omega > \bar{\omega}$, the closed-loop system with states $\tilde{\Gamma}(t)$, $\vartheta(t)$, $u(x,t)$, has a unique exponentially stable periodic solution in t of period $\Pi := 2\pi/\omega$, denoted by $\tilde{\Gamma}^{\Pi}(t), \vartheta^{\Pi}(t),\ u^{\Pi}(x,t)$, satisfying $\forall t \geq 0$:

$$\left(\left|\tilde{\Gamma}^{\Pi}(t)\right|^2 + \left|\vartheta^{\Pi}(t)\right|^2 + \|u^{\Pi}(x,t)\|^2 + \|u_x^{\Pi}(x,t\|^2 + \left|u^{\Pi}(D,t)\right|^2\right)^{1/2} \leq \mathcal{O}\left(1/\omega\right).$$

Furthermore,

$$\limsup_{t\to\infty} |\theta(t) - \theta^*| = \mathcal{O}\left(|a|e^{D\sqrt{\omega/2}} + 1/\omega\right),$$

$$\limsup_{t\to\infty} |\Theta(t) - \Theta^*| = \mathcal{O}\left(|a| + 1/\omega\right),$$

$$\limsup_{t\to\infty} |y(t) - y^*| = \mathcal{O}\left(|a|^2 + 1/\omega^2\right).$$

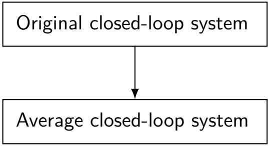
Sketch of Proof

Original closed-loop system

Closed-loop System: $\dot{\tilde{\Gamma}} = \omega_r(\tilde{\Gamma} + H^{-1})[1 - \hat{H}(\tilde{\Gamma} + H^{-1})],$

$$\dot{\vartheta}(t) = u(0,t),$$
$$u_t(x,t) = u_{xx}(x,t), \quad x \in [0,D],$$
$$u_x(0,t) = 0,$$
$$u(D,t) = U(t),$$
$$\dot{U}(t) = -cU(t) - cK\left[z(t) + \int_0^D (D-r)u(r,t)dr\right].$$

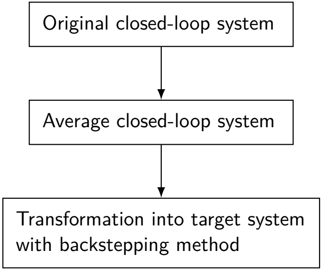
Sketch of Proof



Linearized Average Closed-loop: $\frac{d\tilde{\Gamma}_{\rm av}(t)}{dt} = -\omega_r \tilde{\Gamma}_{\rm av}(t) - \underbrace{\omega_r H \tilde{\Gamma}_{\rm av}^2(t)}_{\text{quadratic}},$

$$
\begin{aligned}
\dot{\vartheta}_{\text{av}}(t) &= u_{\text{av}}(0,t),\\
\partial_t u_{\text{av}}(x,t) &= \partial_{xx} u_{\text{av}}(x,t), \quad x \in [0,\ D],\\
\partial_x u_{\text{av}}(0,t) &= 0,\\
\partial_t u_{\text{av}}(D,t) &= -c u_{\text{av}}(D,t) - cK\left[\vartheta_{\text{av}}(t) + \int_0^D (D-r) u_{\text{av}}(r,t) dr\right].
\end{aligned}
$$

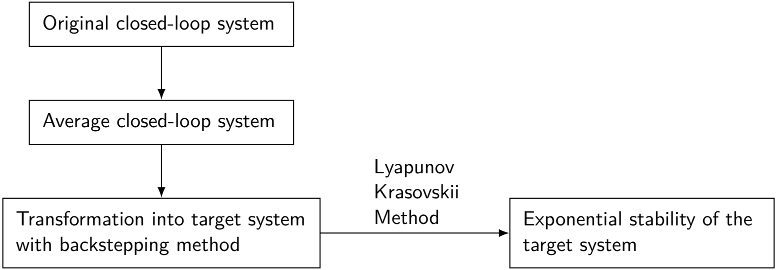
Sketch of Proof



Target system:

$$
\begin{aligned}
\dot{\vartheta}_{av}(t) &= -K\vartheta_{av}(t) + w(0,t), \\
w_t(x,t) &= w_{xx}(x,t) \quad x \in [0,\ D], \\
w_x(0,t) &= 0, \\
w_t(D,t) &= -cw(D,t) + Kw(D,t) \\
&\quad - K^2 \left[\int_0^D \left(e^{-K(D-r)} - 1 \right) w(r,t)dr + e^{-KD}\vartheta_{av}(t) \right].
\end{aligned}
$$

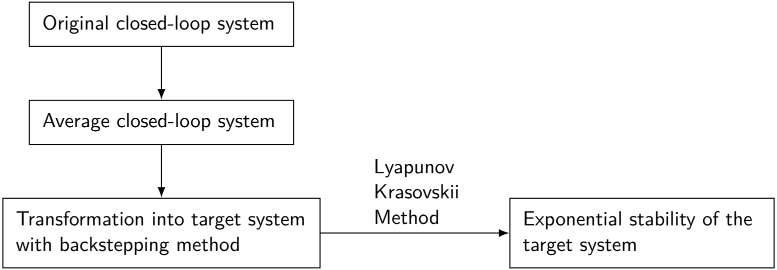
Sketch of Proof



Exponential stability target system:

$$W(t) = \frac{\vartheta_{\mathrm{av}}^2(t)}{2} + \frac{a}{2}\int_0^D w^2(x,t)dx + \frac{b}{2}\int_0^D w_x^2(x,t)dx + \frac{d}{2}w^2(D,t),$$

Sketch of Proof



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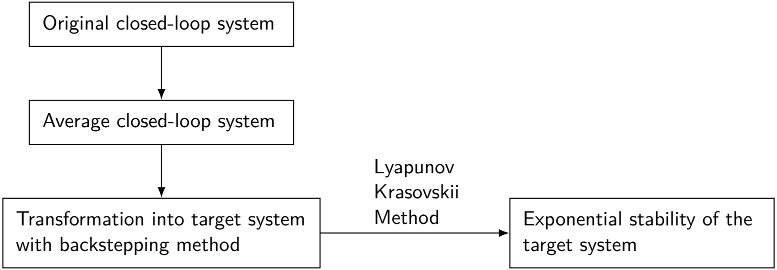
$$\dot{W}(t) \leq -\frac{K}{4}\vartheta_{\mathrm{av}}^2(t) + (c_1^* - c)w^2(D,t) + (c_2^* - c)\|w_x(t)\|^2 - \frac{1}{512D^5K^3}\|w(t)\|^2,$$

Sketch of Proof



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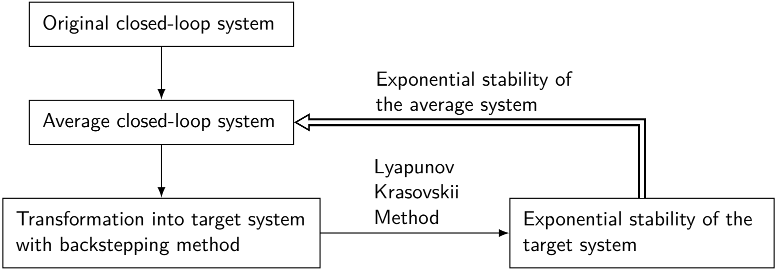
$$\dot{W}(t) \leq -\mu W(t), \quad \mu > 0, \quad c > \max\{c_1^*, c_2^*\}$$

Sketch of Proof

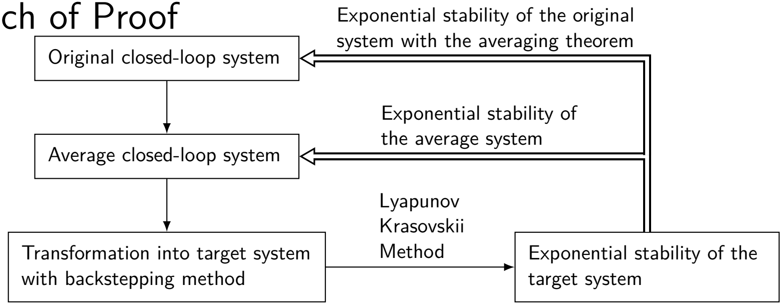


Exponential stability average closed-loop system:

$$\Psi(t) = |\vartheta_{\rm av}(t)|^2 + \int_0^D u_{\rm av}^2(x,t)dx + \int_0^D (u_{\rm av})_x^2(x,t)dx + u_{\rm av}^2(D,t)$$

$$\underline{\rho}\Psi(t) \leq W(t) \leq \bar{\rho}\Psi(t) \quad \Rightarrow \quad \Psi(t) \leq \frac{\bar{\rho}}{\underline{\rho}} e^{-\mu t}\Psi(0)$$

Sketch of Proof

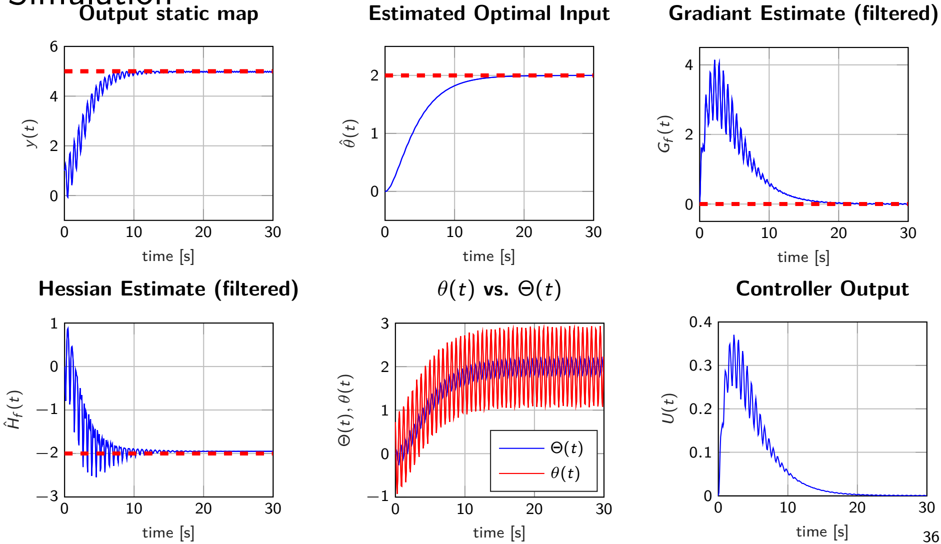


Averaging Theorem[4] (short form)

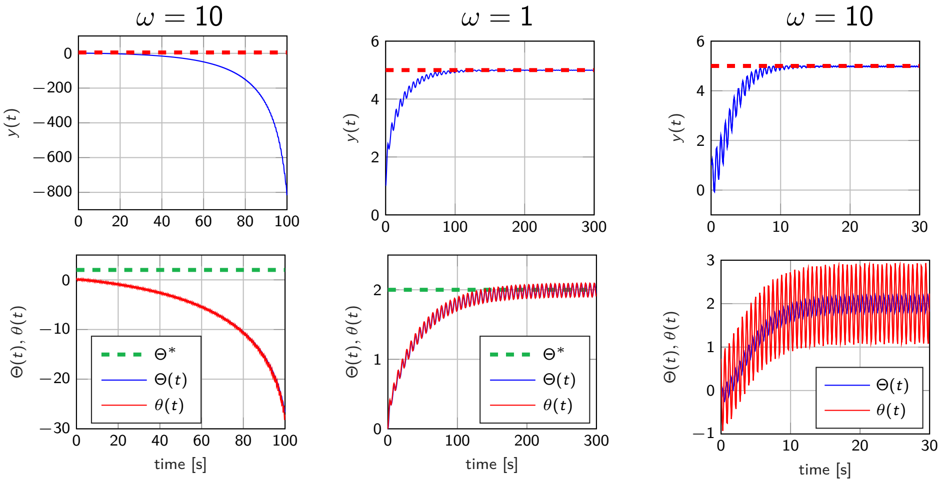
Consider the infinite-dimensional system $\dot{z}(t) = \Gamma z(t) + J(\omega t, z)$ $(*)$, where Γ generates an analytic semigroup and $J(\omega t, z)$ satisfies some smoothness conditions. Then, there exists a periodic solution of $(*)$ $z^{\Pi}(\omega, t)$, with $\|z^{\Pi}\| \leq \mathcal{O}(1/\omega)$, which has the same stability properties as the average solution $z_{\text{av}} = 0$.

[4] J. Hale, S.V. Lunel, et al. "Averaging in infinite dimensions". In: J. Integral Equations Vol. 2.4 (1990), pp. 463–494

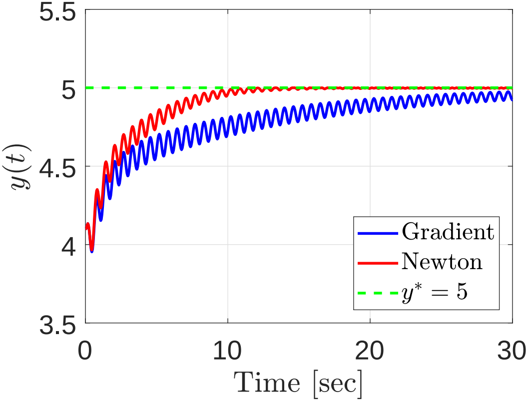
Simulation


Output static map
$y(t)$
time [s]
Estimated Optimal Input
$\hat{\theta}(t)$
time [s]
Gradiant Estimate (filtered)
$G_f(t)$
time [s]
Hessian Estimate (filtered)
$\hat{H}_f(t)$
time [s]
$\theta(t)$ vs. $\Theta(t)$
$\Theta(t), \theta(t)$
$\Theta(t)$
$\theta(t)$
time [s]
Controller Output
$U(t)$
time [s]

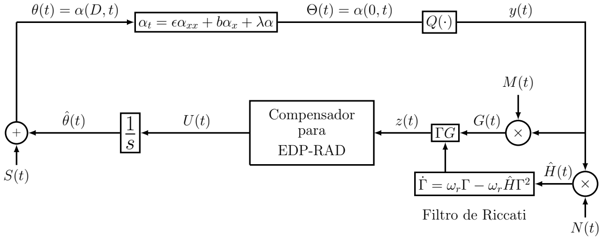
Simulation - basic ES vs. diffusion compensation ES



Simulation - Newton vs. Gradient



PDE Compensation for ES with Reaction-Advection-Diffusion Process



Actuation dynamics (Reaction-Advection-Diffusion):

$\Theta(t) = \alpha(0,t)$

$\alpha_t(x,t) = \epsilon\alpha_{xx}(x,t) + b\alpha_x(x,t) + \lambda\alpha(x,t), \quad x \in [0,1]$

$\alpha(0,t) = 0$

$\alpha(1,t) = \theta(t)$

Output:

$y(t) = Q^{(n)}(\Theta) = y^* + \frac{H}{2}(\Theta(t) - \Theta^*)^2$

Dither Signals:

$S(t) = e^{-\frac{b}{2\epsilon}} \sum_{k=0}^{\infty} \frac{a_{2k}(t)}{(2k)!} + \frac{b}{2\epsilon}\frac{a_{2k}(t)}{(2k+1)!}$

$M(t) = \Upsilon_{n+1}(t)$

$N(t) = \Upsilon_{n+2}(t).$

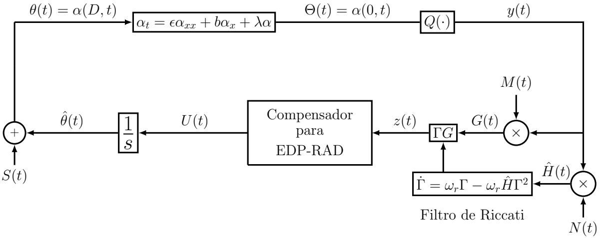
Estimate Error:

$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$

Propagated Estimate Error:

$\vartheta(t) = \hat{\Theta}(t) - \Theta^*$

PDE Compensation for ES with Reaction-Advection-Diffusion Process



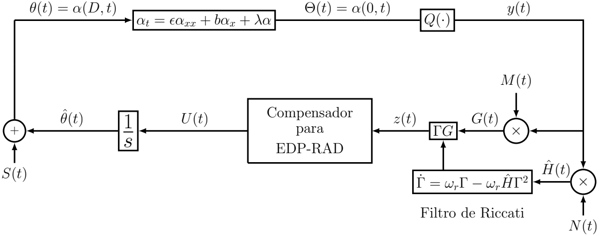
Estimated Error Error Dynamics:

$$\dot{\vartheta}(t) = \partial_x u(0,t),$$
$$u_t(x,t) = \epsilon u_{xx}(x,t) + bu_x(x,t) + \lambda u(x,t), \quad x \in [0,1]$$
$$u_x(0,t) = 0,$$
$$\partial_x u(1,t) = U(t),$$

Control Law with Wave Process Compensation:

$$U(t) = \frac{c}{s+c}\left\{-Ke^{-\frac{b}{2\epsilon}}\left[\bar{\gamma}(1)z(t) + \int_0^1 e^{\frac{b}{2\epsilon}y}\bar{m}(1-y)u(y,t)dy\right]\right\}, K > 0\,.$$

PDE Compensation for ES with Reaction-Advection-Diffusion Process



The **proof of stability and convergence** of the closed-loop system follows the same steps as in Diffusion Process. Nevertheless,

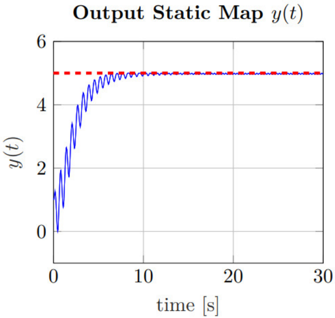
$$\limsup_{t\to\infty} |\theta(t) - \theta^*| = \mathcal{O}\left(|a| \exp\left(\sqrt{\frac{\xi+\omega}{\epsilon}}\right) + \frac{1}{\omega}\right),$$

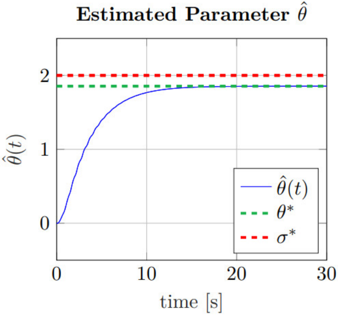
$$\limsup_{t\to\infty} |\Theta(t) - \Theta^*| = \mathcal{O}\left(|a| + \frac{1}{\omega}\right) \text{ and}$$

$$\limsup_{t\to\infty} |y(t) - y^*| = \mathcal{O}\left(|a|^2 + \frac{1}{\omega^2}\right).$$

PDE Compensation for ES with Reaction-Advection-Diffusion Process: Simulation

$H = -2,\ \Theta^* = 2,\ y^* = 5,\ n = 1,\ \omega = 10,\ a = 0.2,\ c = 20,$
$K = 0.1,\ \epsilon = 1,\ b = 1,\ \lambda = 0.2$ and $K = 0.4$





PDE Compensation for ES with Reaction-Advection-Diffusion Process: Simulation

$H = -2,\ \Theta^* = 2,\ y^* = 5,\ n = 1,\ \omega = 10,\ a = 0.2,\ c = 20,$
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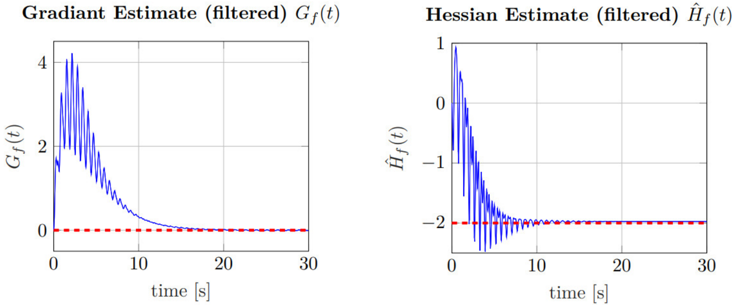
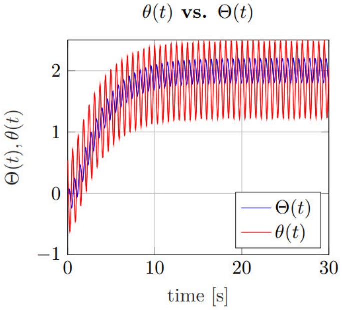
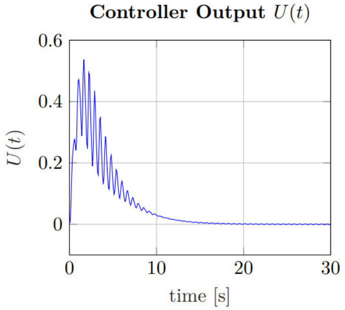


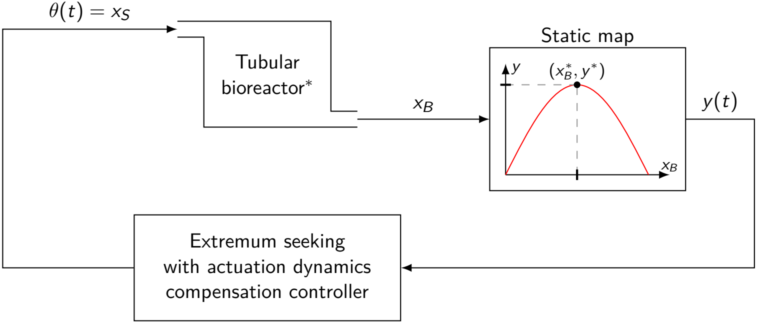
PDE Compensation for ES with Reaction-Advection-Diffusion Process: Simulation

$H = -2,\ \Theta^* = 2,\ y^* = 5,\ n = 1,\ \omega = 10,\ a = 0.2,\ c = 20,$
$K = 0.1,\ \epsilon = 1,\ b = 1,\ \lambda = 0.2$ and $K = 0.4$





Application



System

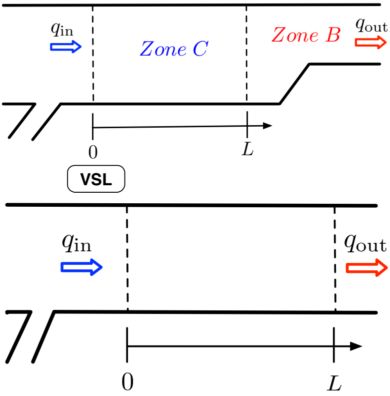
- x_S : Substrate concentration
- x_B : Biomass concentration

Goal

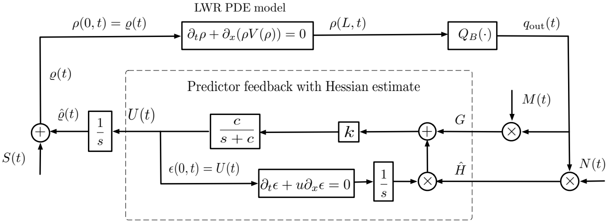
- Find and maintain the optimal product rate

* JJ. Winkin, D. Dochain, and P. Ligarius. "Dynamical analysis of distributed parameter tubular reactors". In: Automatica 36.3 (2000), pp. 349–361.

Predictor Feedback for ESC with LWR Process - Application to **Traffic Control**



Predictor Feedback for ESC with Lighthill-Whitham-Richards (LWR) Process



Actuation dynamics (LWR):

$\partial_t \rho + \partial_x(Q_C(\rho)) = 0$ where
$x \in [0, L],\ t \in [0, \infty)$

$Q_C(\rho) = -\frac{v_f}{\rho_m}\rho^2 + v_f \rho$

Boundary flow:

$q_{\text{in}}(t) = Q_C(\rho(0,t))$

$q_{\text{out}}(t) = Q(\rho(L,t))$

Locally:

$q_{\text{out}}(t) = q^\star + \frac{H}{2}\left(\varrho\left(t - D\right) - \rho^\star\right)^2$

Dither Signals:

$S(t) = a \sin\left(\omega(t + D)\right)$

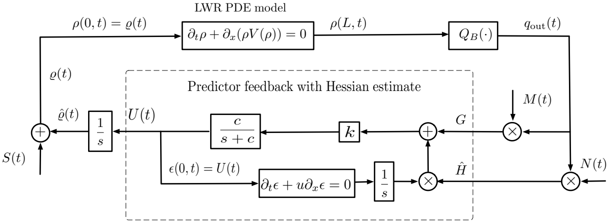
$M(t) = \frac{2}{a}\sin\left(\omega t\right)$

$N(t) = -\frac{8}{a^2}\cos\left(2\omega t\right).$

Estimate Error:

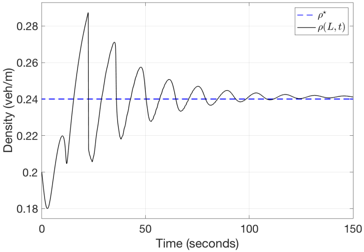
$e(t) = \hat{\varrho}(t) - \rho^\star$

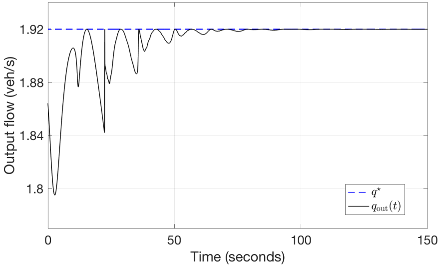
Predictor Feedback for ESC with Lighthill-Whitham-Richards (LWR) Process



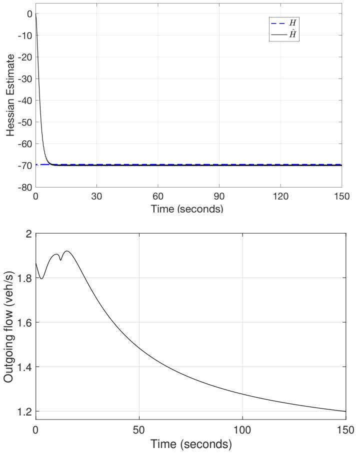
Theorem: Consider the closed-loop system. There exits $c_0>0$ such that $\forall c\geq c_0$, there exists $\omega_0(c_0)>0$ such that $\forall\omega>\omega_0$, the closed-loop system has a unique exponentially stable periodic solution in period $T=\frac{2\pi}{\omega}$, denoted by $e^T(t-D), U^T(\tau), \forall\tau\in[t-D,t]$, satisfying $\forall t>0$ $\left(|e^T(t-D)|^2+|U^T(t)|^2+\int_0^D|U^T(\tau)|^2d\tau\right)^{\frac{1}{2}}\leq\mathcal{O}(1/\omega)$. Furthermore, $\lim_{t\to+\infty}\sup|\varrho(t)-\rho^\star|=\mathcal{O}(a+1/\omega)$ and $\lim_{t\to+\infty}\sup|q_{\text{out}}(t)-q^\star|=\mathcal{O}(a^2+1/\omega^2)$.

Predictor Feedback for ESC with LWR Process - Simulation

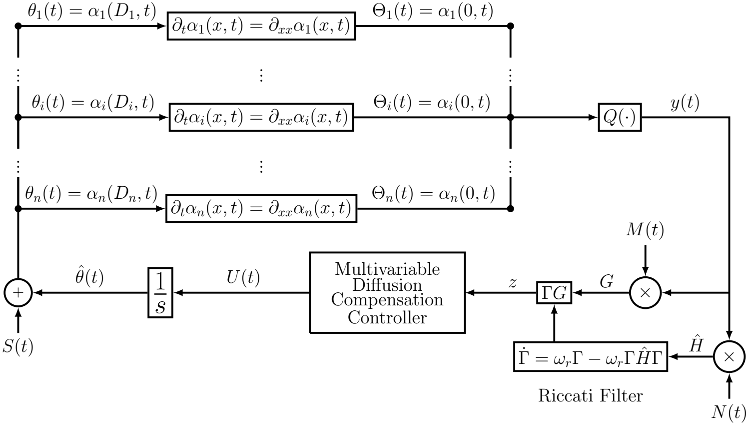




Predictor Feedback for ESC with LWR Process - Application



Challenge - Multivariable Newton-based ES

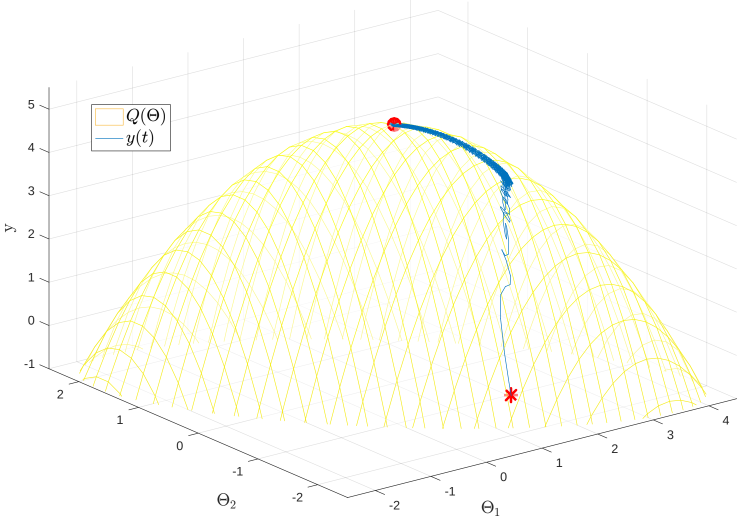


The Newton algorithm effectively “diagonalizes” the map and allows “decentralized” compensators for each control channel, whereas the Gradient algorithm has to perform diffusion compensation of the cross-coupling of the channels.

Multivariable ES for Distinct Classes of PDE Systems

RAD Equation	PDE : $\partial_t\alpha_i(x,t)=\epsilon_i\partial_{xx}\alpha_i(x,t)+b_i\partial_x\alpha_i(x,t)+\lambda_i\alpha_i(x,t)\,,\quad \epsilon_i>0\,,b_i\geq 0\,,\lambda_i\geq 0$ Boundary Control (Dirichlet) : $U_i(t)=\frac{c_i}{s+c_i}\left\{-k_ie^{-\frac{b_i}{2\epsilon_i}}\left[\gamma(1)z_i(t)+\int_0^1 e^{\frac{b_i}{2\epsilon_i}\sigma}m(1-\sigma)u(\sigma,t)d\sigma\right]\right\},$ $\gamma(x)=\cosh\left(\sqrt{\frac{\xi}{\epsilon_i}}x\right)+\frac{b_i}{2\epsilon_i}\sqrt{\frac{\epsilon_i}{\xi}}\sinh\left(\sqrt{\frac{\xi}{\epsilon_i}}x\right),\quad \xi:=b_i^2/(4\epsilon_i)-\lambda_i\geq 0,$ $m(x-\sigma)=\frac{1}{\epsilon_i}\sqrt{\frac{\epsilon_i}{\xi}}\sinh\left(\sqrt{\frac{\xi}{\epsilon_i}}(x-\sigma)\right),\quad x\in[0,1]$ Trajectory Generation : $S_i(t)=e^{-\frac{b_i}{2\epsilon_i}}\sum_{k=0}^{\infty}\frac{a_{2k}(t)}{(2k)!}+\frac{b_i}{2\epsilon_i}\frac{a_{2k}(t)}{(2k+1)!},$ $a_{2k}:=\frac{a_i}{\epsilon_i^k}\sin(\omega_i t)\sum_{n=0}^{k}\binom{k}{2n}\xi^{k-2n}\omega_i^{2n}+\frac{a_i}{\epsilon_i^k}\cos(\omega_i t)\sum_{n=0}^{k}\binom{k}{2n+1}\xi^{k-2n-1}\omega_i^{2n+1}$
Wave Dynamics	PDE : $\partial_{tt}\alpha_i(x,t)=\partial_{xx}\alpha_i(x,t),\quad x\in[0,D_i]$ Boundary Control (Neumann) : $U_i(t)=\frac{c_i}{s+c_i}\left\{c\left[-k_iu_i(D_i,t)-\partial_t u_i(D_i,t)\right]+\rho(D_i)z_i+\right.$ $\left.\int_0^{D_i}\rho(D_i-\sigma)\partial_t u_i(\sigma,t)d\sigma\right\},\quad \rho(s)=-k_i[0\ \ I]e^{\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}s}[0\ \ I]^T$ Trajectory Generation : $S_i(t)=\frac{a_i}{\omega_i}\sin(\omega_iD_i)\sin(\omega_it)$
Delays	PDE : $\partial_t\alpha_i(x,t)=\partial_x\alpha_i(x,t),\quad x\in[0,D_i]$ Boundary Control (Dirichlet) : $U_i(t)=\frac{c_i}{s+c_i}\left\{-k_i\left[z_i(t)+\int_0^{D_i}u_i(\sigma,t)d\sigma\right]\right\}$ Trajectory Generation : $S_i(t)=a_i\sin(\omega_i(t+D_i))$

Challenge - Multivariable Newton-based ES $(i = 1, 2)$



Conclusion

Assumptions

- known actuator dynamics
- unknown static map
- existence of extremum

Results

- semi-model based
- exponential stability
- local convergence
- convergence speed independent of the Hessian

Extensions

- multivariable Newton-based ESC
- dynamic plants (ODE+PDE)
- measurement dynamics described by diffusion PDEs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