Synchronization and Applications.

Prof. Hildebrando Munhoz Rodrigues.
Instituto de Ciências Matemáticas e de Computação. USP. São Carlos

DINCON 2019, 27 de novembro de 2019.

SYNCHRONIZATION

- **MECHANICAL OSCILLATORS**
- **ELETRONIC CIRCUITS** Chua, Koçarev, Belykh & Verichev
- **TELECOMMUNICATIONS** Cuomo-Oppenheim & C. Tresse
- **BIOLOGICAL OSCILLATORS** Mirolo & Strogatz
- **COUPLED LASERS**, Raj Roy and his group. University of Maryland.
- **CHEMICAL OSCILLATORS** Belousov & Zhabotinski
- **MENSTRUAL CYCLE**
- **FIREFLIERS**
- **POWER SYSTEMS**
- **NEUROBIOLOGY**- Hodgkin-Huxley Equations
- **EPIDEMIOLOGICAL MODELS** - Chagas deasese, Dengue

Collaborators.

- Valentin Afraimovich
- Isabel Labouriau (Hodgkin-Huxley)
- Luis Fernando Costa Alberto (PHd, Power Systems.)
- Marcio Gameiro (Master) (Continuous. Communication.)
- L. R. A. Gabriel Filho (Master) Jianhong Wu. (Discrete)
- Wescley Bonomo (Master) (Discrete)
- Marcio Gameiro, Wu (Discrete, Communication.)
- Marcio Gameiro, Wu (Continuous, Switching.)
- Alexandre Nolasco de Carvalho (PDEs)

More References..

- Afraimovich, V. S. & Rodrigues, H. M. [1998] "Uniform Dissipativeness and Synchronization on Nonautonomous Equations,", Equadiff95, International Conference on Differential, World Scientific, 3-17.
- Carvalho, A. N., Dlotko, T. & Rodrigues, H. M.[1998] "Upper Semicontinuity of attractors and synchronization,"Journal of Mathematical Analysis and Applications, **220**, 13-41.
- Gameiro, M. F. & Rodrigues, H. M. [2001] "Applications of Robust Synchronization to Communication Systems,"Applicable Analysis, **79**, 21-45.
- Labouriau, I. S. & Rodrigues, H. M.[2003] "Synchronization of coupled equations of Hodgkin-Huxley type,"Dynamics of Continuous, Discrete and Impulsive Systems. Ser. A., **10**, 463-476.
- Rodrigues, H. M.[1996] "Abstract Methods for Synchronization and Applications,"Applicable Analysis, **62**, 263-296.
- Rodrigues, H. M., Alberto, L. F. C. & Bretas, N. C.[2000] "On the Invariance Principle. Generalizations and Aplications to Synchronism,"IEEE Transactions on Circuit ans Systems, IEEE Transactions on Circuit ans Systems-I: Fundamental Theory and Applications, **47**:5, 730-739.
- Rodrigues, H. M., Alberto, L. F. C. & Bretas, N. C. [2001] "Uniform invariance principle and synchronization, robustness with respect to parameter variation", Journal of Differential Equations, **169**:1, 228-254.
- Rodrigues, H. M., Wu, J. & Gameiro, "Robust Synchronization of Parametrized Nonautonomous Discrete Systems with Applications to Communication Systems"Journal of Applied Analysis and Computation, v. 1, p. 537-547, 2011.
- Rodrigues, H. M., Wu, J. & Gabriel, L. R. A. Uniform Dissipativeness, Robust Synchronization and Location of the Attractor of Parametrized Nonautonomous Discrete Systems, Int. J. of Bif. and Chaos, Vol. 21, No. 2 (2011) 513 to 526.

Invariance Principle and Uniform Invariance Principle.

Invariance Principle.
J. P. LaSALLLE,
"*Some Extensions of Liapunov's Second Method", IRE Transactions on Circuit Theory, p. 520-527, (1960).*
$\dot{x} = f(x), V(x), \dot{V}(x) \leq 0$

Uniform Invariance Principle.
H. M. Rodrigues, L. F.C. Alberto and N. G. Bretas,
Uniform Invariance Principle and Synchronization, Robustness with respect to Parameter Variation. *J. Differential Equations* **169** *228-254 (2001).*
$\dot{x} = f(x, \lambda), V(x, \lambda),\ \dot{V}(x, \lambda)$ *may change sign.*

LaSalle Invariance Principle

Some Extensions of Liapunov's Second Method*

J. P. LASALLE†

Introduction

LIAPUNOV'S second method has long been recognized in the Soviet Union as the most general method for the study of the stability of equlibirum positions of systems described by differential or difference equations. The method was first presented by Liapunov in his now classical memoir,[1] which appeared in Russian in 1892 and was translated into French in 1907. Good sources for the statements and proofs of the mathematical theorems underlying the method can be found in works by Hahn,[2] Antosiewicz,[3] and Cesari.[4] These references also contain extensive bibliographies.

By way of introduction, let us consider first Liapunov's asymptotic stability theorem for autonomous systems. Let the systems of differential equations be ($\dot{x} = dx/dt$)

$$\dot{x}_i = X_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n. \tag{1}$$

The state of the system at time t is given by n real numbers $x_1(t), x_2(t), \cdots, x_n(t)$. Thus, the state of the system at time t can be represented simply by the n vector $x(t) = (x_1(t), \cdots, x_n(t))$. The phase velocity of the system at the point $x = (x_1, \cdots, x_n)$ is defined by the vector field $X(x) = (X_1(x), \cdots, X_n(x))$. In vector notation, the system of differential equations is simply the vector differential equation

$$\dot{x} = X(x). \tag{2}$$

The objective of this paper is to present methods rather than to obtain general results, and we shall confine our illustrations to simple equations such as Liénard's:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{3}$$

Letting $F(x) = \int_0^x f(u)du$ and $y = \dot{x} + F(x)$, we obtain, as a convenient system of first-order equations equivalent to (3), the system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x). \tag{4}$$

For van der Pol's equation $f(x) = \epsilon(x^2 - 1)$, $F(x) = \epsilon(\frac{1}{3}x^3 - x)$, and $g(x) = x$.

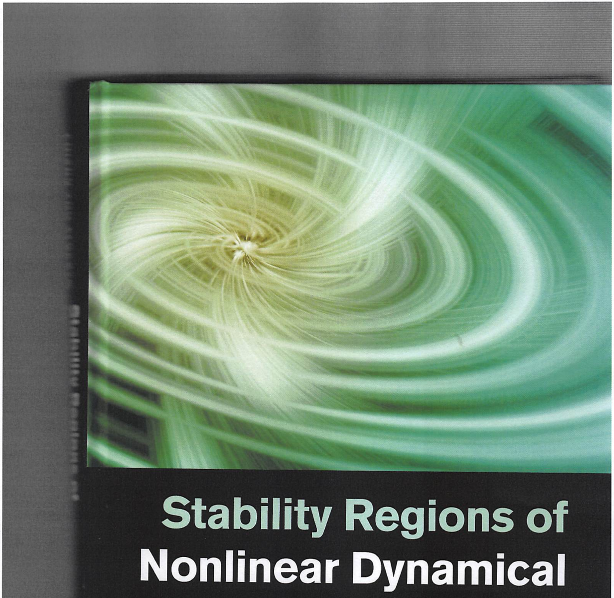
Returning to the general system (2), we shall assume that $X(x)$ has continuous first partials for all x. Thus, for any x^0 there is a unique solution $x(t)$ of (2) satisfying $x(0) = x^0$. This assumption on $X(x)$ is much stronger than is required, but this is of little concern to us here, and our attitude throughout is to present the principal features of methods. The *equilibrium states*, sometimes called critical points, are those states where $X(x) = 0$. Thus, if $X(x^0) = 0$, then $x = x^0$ is a solution of (2). Started at x^0, the system remains in this state for all t. This is, of course, a mathematical statement, and the actual behavior of physical systems raises the problem of stability.

It is never possible to start the system exactly in its equilibrium state, and the system is always subject to outside forces not taken into account by the differential equations. The system is disturbed and is displaced slightly from its equilibrium state. What happens? Does it remain near the equilibrium state? This is stability. Does it remain near the equilibrium state and in addition tend to return to the equilibrium? This is asymptotic stability.

Let us make these notions more precise. Assume, as one always can, that the equilibrium state being investigated is located at the origin: $X(0) = 0$. A translation of coordinates accomplishes this. Let $||x||$ be the Euclidean length of the vector x: $||x||^2 = x_1^2 + \cdots + x_n^2$. Let $S(R)$ be the spherical region of radius $R > 0$ about the origin: $S(R)$ consists of the points x satisfying $||x|| < R$. The origin is said to be *stable* if corresponding to each $S(R)$ there is an $S(r)$, such that a solution starting in $S(r)$ does not leave $S(R)$: $x(0)$ in $S(r)$ implies $x(t)$ is in $S(R)$ for all $t \geq 0$ (Fig 1). If, in addition, there is a neighborhood $S(R_0)$ such that every solution starting in $S(R_0)$ approaches the origin as $t \to \infty$, the system is said to be *asymptotically stable* (Fig. 2).

For instance, if within a neighborhood of the equi-

Book by H. Chiang and L. F. C. Alberto.



Lorenz System.

$$\begin{cases} \dot{x} = -\sigma\ x + \sigma\ y \\ \dot{y} = r\ x - y - x\ z \\ \dot{z} = -b\ z + x\ y \end{cases}$$

Reference Values: $\sigma = 10,\ r = 28,\ b = 8/3.$

Lorenz System. Master-slave.

$$
\begin{cases}
\dot{x} = -\sigma x + \sigma y \\
\dot{y} = r\,(x + m(t)) - y - (x + m(t))\,z \\
\dot{z} = -bz + (x + m(t))\,y \\
\\
\downarrow \quad x(t) + m(t) \\
\\
\dot{u} = -\sigma u + \sigma v \\
\dot{v} = r\,(x(t) + m(t)) - v - (x(t) + m(t))\,w \\
\dot{w} = -bw + (x(t) + m(t))\,v
\end{cases}
\tag{1}
$$

Lorenz. Synchronization Master-slave

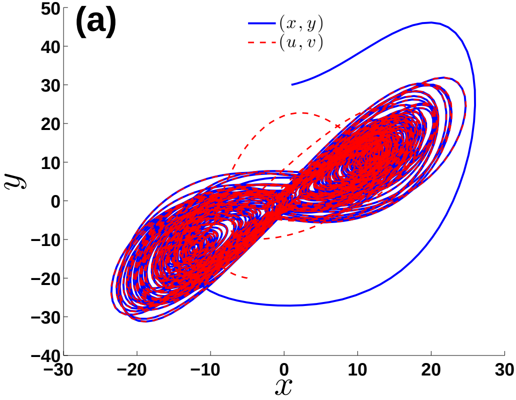


Figura: synchronization

Lorenz error

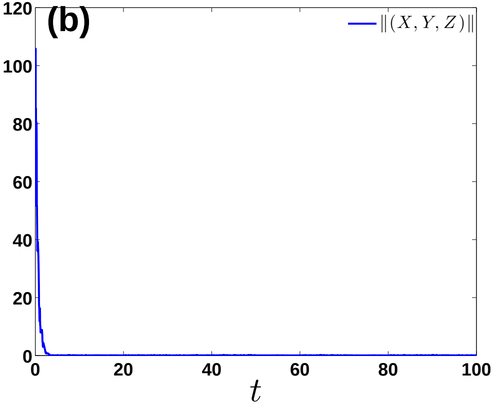


Figura: difference between master and slave solutions

Lorenz. Original and Codified Signals

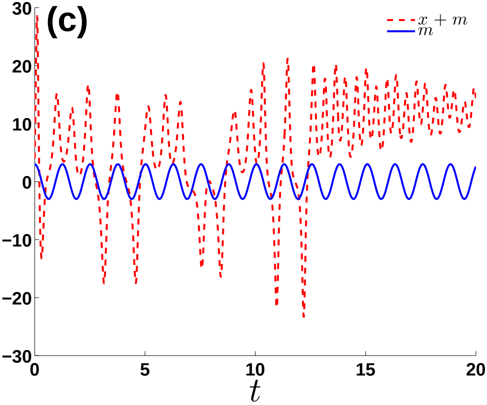


Figura: original and codified signals

Lorenz. Original and Decodified Signals.

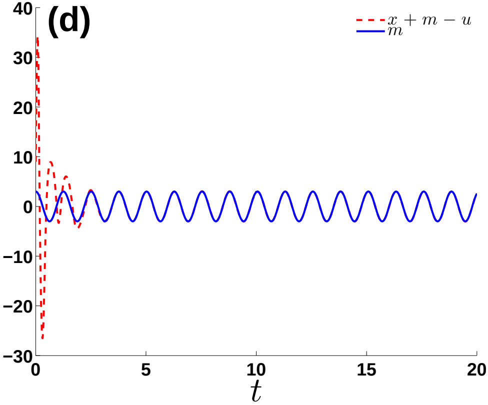


Figura: original and recovered signals

Chua System.

$$\begin{cases} \dot{x} = -\alpha x + \alpha y - \alpha \ h(x, a, b) \\ \dot{y} = -x - y + z \\ \dot{z} = -\beta y - \sigma z \end{cases}$$

$$h(x, a, b) = bx + \frac{(a-b)}{2}\left(|x+1| - |x-1|\right). \ a > b \text{ and } b < 1.$$

Reference values $\alpha = 7, \beta = 100, \ a = \frac{8}{7}, \ \sigma = \frac{1}{2}, \ b = \frac{5}{7}$.

Master-Slave: Chua

$$\begin{cases} \dot{x} = -\alpha x + \alpha y - \alpha \; h\,(x + m(t), a, b) \\ \dot{y} = -\,(x + m(t)) - y + z \\ \dot{z} = -\beta y - \sigma z \\ \\ \downarrow \quad x(t) + m(t) \\ \\ \dot{u} = -\alpha u + \alpha v - \alpha \; h\,(x(t) + m(t), a, b) \\ \dot{v} = -\,(x(t) + m(t)) - v + w \\ \dot{w} = -\beta v - \sigma w \end{cases} \tag{2}$$

chua synchronization

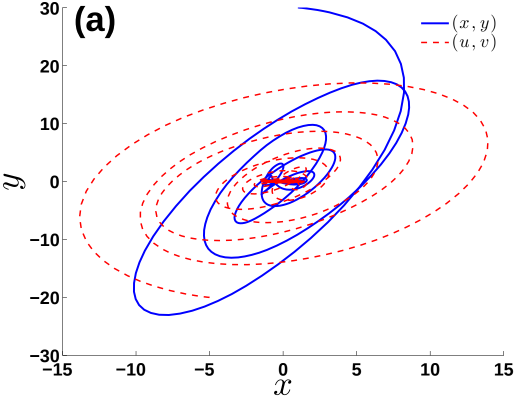


Figura: chua synchronization

chua original and codified signals

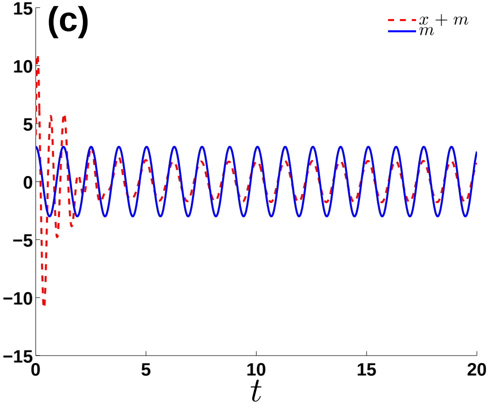


Figura: chua original and codified signals

chua original and decodified signals

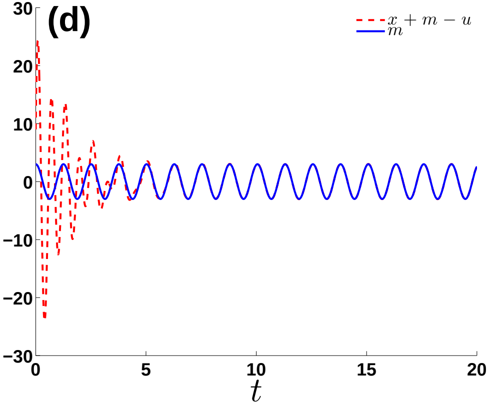


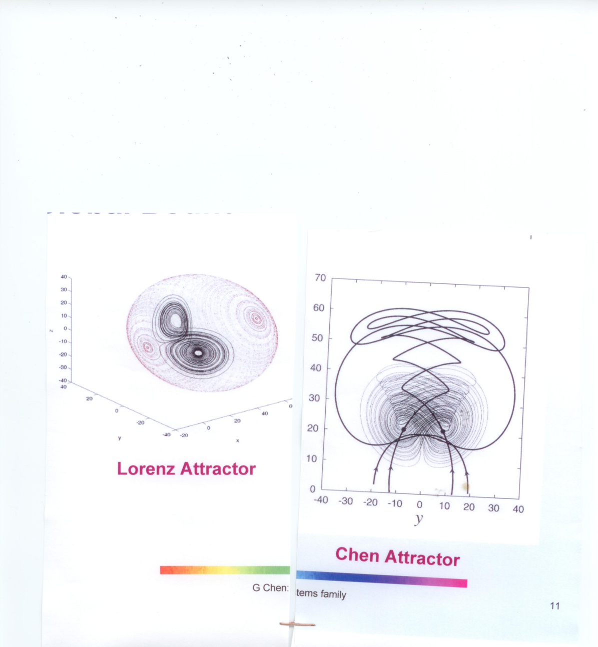
Figura: chua original and decodified signals

chen system

$$\begin{cases} \dot{x} = -ax + ay \\ \dot{y} = -ax + cx + cy - xz \\ \dot{z} = -bz + xy \end{cases}$$

Reference values $a = 35, \; b = 3, \; c = 28.$

lorenz-chen



chen master-slave systems

$$
\begin{cases}
\dot{x} = -ax + ay \\
\dot{y} = (c-a)x + (c-a)y - (x + m(t))\, z + a(y + \ell(t)) \\
\dot{z} = -bz + (x + m(t))\, y \\
\\
\downarrow \quad x(t) + m(t) \qquad \downarrow \quad y(t) + \ell(t) \\
\\
\dot{u} = -au + av \\
\dot{v} = (c-a)u + (c-a)v - (x(t) + m(t))\, w + a(y(t) + \ell(t)) \\
\dot{w} = -bw + (x(t) + m(t))\, v
\end{cases}
\tag{3}
$$

chen synchronization

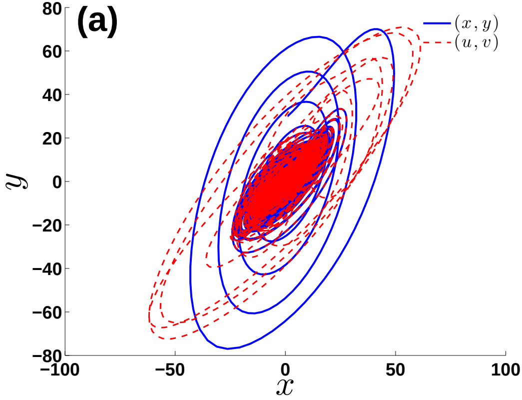


Figura: chen synchronization

chen original and codified signals

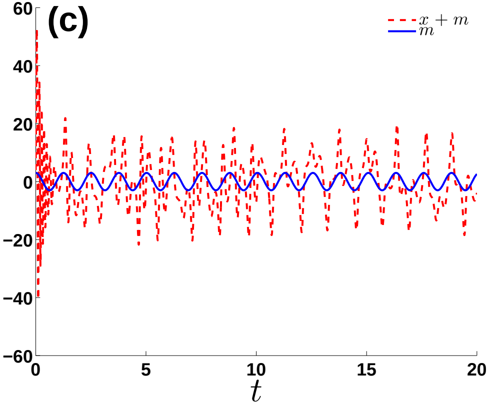


Figura: chen original and codified signals

chen original and recovered signals

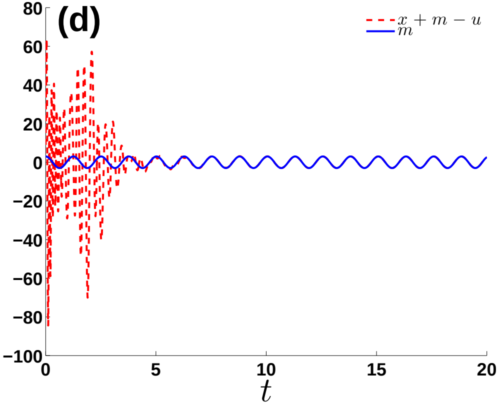


Figura: chen original and recovered signals

Communication via switching systems

$$Master - Slave1 \begin{cases} \dot{x} = -\sigma_1 x + \sigma_1 y \\ \dot{y} = r_1 (x + m(t)) - y - (x + m(t)) z \\ \dot{z} = -b_1 z + (x + m(t)) y \\ \downarrow \quad x(t) + m(t) \\ \dot{u} = -\sigma_1 u + \sigma_1 v \\ \dot{v} = r_1 (x(t) + m(t)) - v - (x(t) + m(t)) w \\ \dot{w} = -b_1 w + (x(t) + m(t)) v \end{cases} \tag{4}$$

$$Master - Slave2 \begin{cases} \dot{x} = -\sigma_2 x + \sigma_2 y \\ \dot{y} = r_2 (x + m(t)) - y - (x + m(t)) z \\ \dot{z} = -b_2 z + (x + m(t)) y \\ \downarrow \quad x(t) + m(t) \\ \dot{u} = -\sigma_2 u + \sigma_2 v \\ \dot{v} = r_2 (x + m(t)) - v - (x + m(t)) w \\ \dot{w} = -b_2 w + (x + m(t)) v \end{cases} \tag{5}$$

Global Dissipativeness. Typical result.

$$\dot{x} = f(x, \lambda), \ x \in \mathbb{R}^n, \quad \lambda \in \Lambda \tag{6}$$

$$a, \ b, \ c \in C(\mathbb{R}^n, \mathbb{R}), \ a(x) \to \infty, \ \|x\| \to \infty$$

$$a(x) \leq V(x, \lambda)) \leq b(x), \ -\dot{V}(x, \lambda) \geq c(x)$$

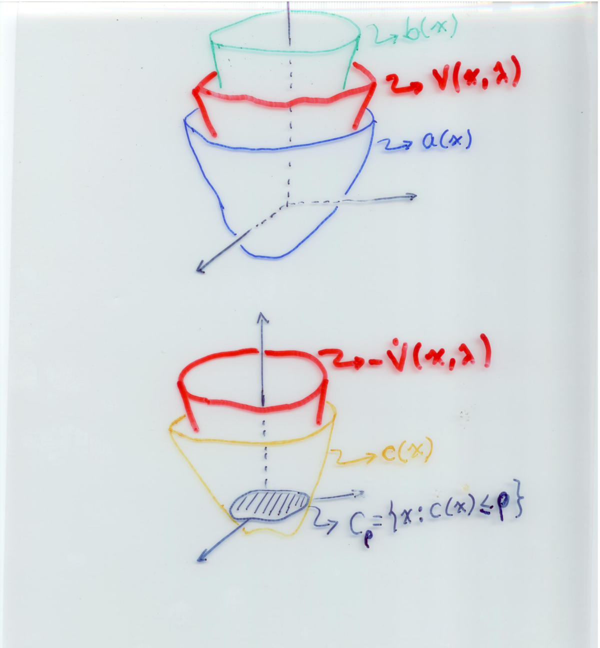
$$\exists \ \rho > o : C_\rho := \{x \in \mathbb{R}^n : c(x) \leq \rho\} \neq \emptyset \ \text{ and bounded}$$

$$r > \sup_{x \in C_\rho} b(x), \quad \mathcal{A}_r := \{x \in \mathbb{R}^n : a(x) \leq r\}$$

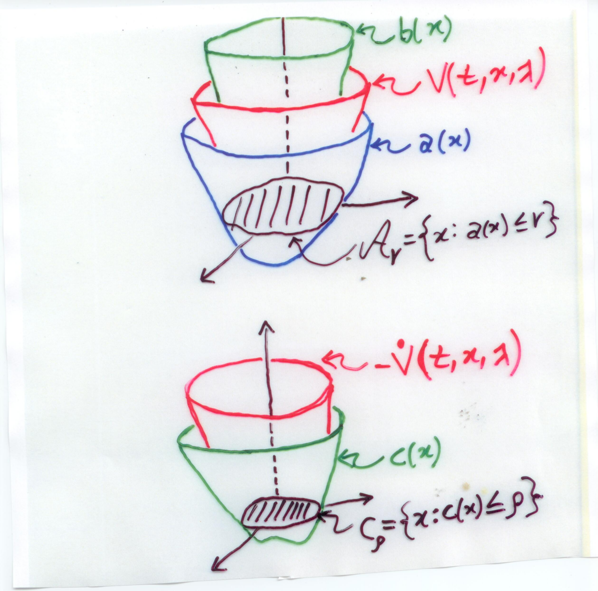
Then

- If $x(t, t_0, x_0)$ is a solution of (6)
  $\exists \ t_1 \geq t_0 : \ x(t, t_0, x_0) \in \mathcal{A}_r, \ t \geq t_1$
- If $x(t)$ is a solution fo (6) bounded for $\forall t \in \mathbb{R}$ then
  $x(t) \in \mathcal{A}_r, \quad \forall t \in \mathbb{R}$.

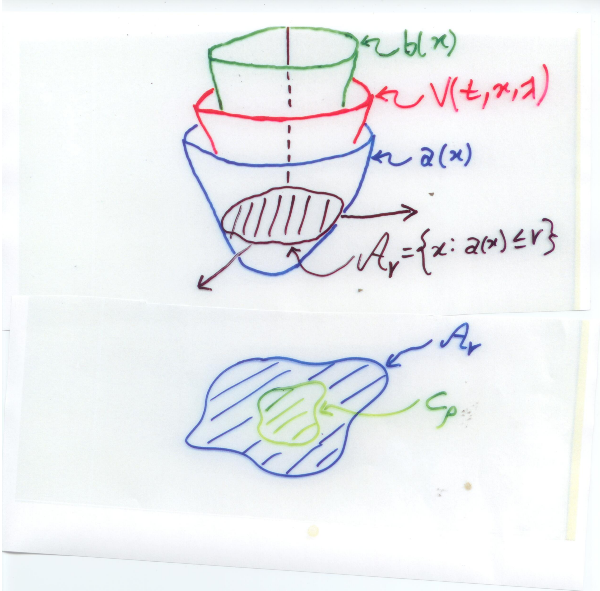
Estimate of the attractor. Liapunov functions



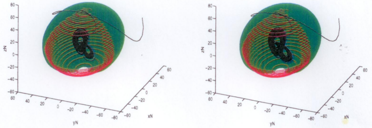
Estimates.



Estimates.



Estimate of the lorenz attractor.



(a) $\sigma = 9.5, r = 26.6, b = 2.53$

(b) $\sigma = 10.5, r = 29.4, b = 2.8$

Figure 1: **Lorenz System**

Estimate of the lorenz attractor.

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = r\left(x + m(t)\right) - y - \left(x + m(t)\right) z \\ \dot{z} = -bz + \left(x + m(t)\right) y \\ \dot{u} = -\sigma u + \sigma v \\ \dot{v} = r\left(x + m(t)\right) - v - \left(x + m(t)\right) w \\ \dot{w} = -bw + \left(x + m(t)\right) v \end{cases} \tag{7}$$

Let $X = x - u, \quad Y = y - v, \quad Z = z - w$

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = -Y - \left(x + m(t)\right) Z \\ \dot{Z} = -b\ Z + \left(x + m(t)\right) Y \end{cases} \tag{8}$$

$$(X(t), Y(t), Z(t)) \ \to 0 \ \text{ as } \ t \ \to \ \infty.$$

Liapunov function.

$$V(X, Y, Z) = \frac{1}{2}(X^2 + \sigma Y^2 + \sigma Z^2)$$

$$-\dot{V} = \sigma X^2 - \sigma XY + \sigma Y^2 + \sigma b Z^2.$$

Thank You!